B.Sc. (Maths) part II
paper – III

Topic: Monotonic Sequence

Def: – A sequence $\{a_n\}$ is said to be monotonic if it is either monotonic increasing or monotonic decreasing

Monotonic increasing

Def: – A sequence $\{a_n\}$ is said to be monotonic increasing if $a_{n+1} \geq a_n$ $\forall n \in N$

Ex: – $a_n = n^2$

Def: – A sequence $\{a_n\}$ is said to be monotonic decreasing if $a_{n+1} \leq a_n$

Ex: – $a_n = -n^2$

Remarks: – (i) Every sequence not be monotonic

$\{(-1)^n\} = \{-1, 1, -1, 1 \ldots\}$ is neither increasing nor decreasing

(ii) Constant ~~function~~ sequence is both monotonic increasing and monotonic decreasing sequence.

Theorem (1) Every monotonic increasing sequence tend to its least upper bound.

Proof :- Let $\{a_n\}$ be monotonic increasing sequence whose least upper bound is M. Then by def of upper bound

$a_n \le M \; \forall$ values of n. — (1)

and $a_n + \epsilon > M$ for at least one value of n

Let this be true for $n = m$

Then $a_m + \epsilon > M$ ie $a_m > M - \epsilon$ — (2)

Since the sequence is monotonic increasing therefore

$a_n \ge a_m$ when $n \ge m$ — (3)

From (2) and (3) $a_n > M - \epsilon \; \forall \; n \ge m$

Again from (1) $a_n \le M$ for all n

So, $a_n < M + \epsilon \; \forall \; n$ and hence also for $n \ge m$

Combining these result we obtain

$M - \epsilon < a_n < M + \epsilon$ for $n \ge m$

$\therefore |a_n - M| < \epsilon$ for $n \ge m$

Hence $\{a_n\}$ tends to M

Hence the theorem

Theorem (2) Every monotonic decreasing sequence tends to its greatest lower bound.

Proof :- Let $\{a_n\}$ be a monotonic decreasing sequence whose greatest lower bound is K

Then by def of lower bound

$a_n \geq K$ for all values of n

and $a_n < K + \epsilon$ for at least one value of n — (1)

Let it is true for $n = m$

Then $a_m < K + \epsilon$ — (2)

Since $\{a_n\}$ is monotonic decreasing sequence therefore

$a_n \leq a_m$ for $n \geq m$ — (3)

From (2) and (3) $a_n < K + \epsilon$ for $n \geq m$

Again from (1) $a_n > K - \epsilon$ for n and so for $n \geq m$

Combining these result we obtain

$K - \epsilon < a_n < K + \epsilon$ for $n \geq m$

i.e. $|a_n - K| < \epsilon$, for all $n \geq m$

Hence $\{a_n\}$ tend to greatest lower bound K

Theorem :- Show that the necessary and sufficient condition for the converge of a monotonic sequence is that bounded.

Proof :- The condition is necessary

Let us suppose that a monotonic sequence $\{a_n\}$ is convergent. It is required to prove that $\{a_n\}$ is bounded.

This follow from the fact that every convergent sequence is bounded.

This condition is sufficient

We want to prove that monotonic sequence which is bounded is convergent.

This follow from use of Theorem (1) and Theorem (2).